ON THE MEMBRANE THEORY OF ANISOTROPIC SHELLS

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The question of constructing a membrane theory of anisotropic shells by reducing the three-dimensional problem of the theory of elasticity of an anisotropic body to a two-dimensional problem of shell theory is considered. But special attention is hence turned to those stresses which are not the subject of discussion in the classical theory of anisotropic shells [1, 2]. Some analytical criteria for a shell to be a membrane are proposed.

1. Let us consider a homogeneous anisotropic shell of constant thickness h, when there is just one plane of elastic symmetry, parallel to the shell middle surface, at each point of the shell. It is considered that the middle surface is represented by orthogonal curvilinear coordinates α, β, which coincide with the lines of principal curvature of the middle surface. The position of any point on the shell is defined by the orthogonal coordinates α, β, γ, where the coordinate γ is rectilinear. It is also considered that the shell remains elastic, subject during deformation to a generalized Hooke's law of the form [1]

$$\begin{gathered}
e_\alpha = a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{13}\sigma_\gamma + a_{16}\tau_{\alpha\beta}, \quad e_{\beta\gamma} = a_{44}\tau_{\beta\gamma} + a_{45}\tau_{\alpha\gamma} \\
e_\beta = a_{12}\sigma_\alpha + a_{22}\sigma_\beta + a_{23}\sigma_\gamma + a_{26}\tau_{\alpha\beta}, \quad e_{\alpha\gamma} = a_{45}\tau_{\beta\gamma} + a_{55}\tau_{\alpha\gamma} \\
e_\gamma = a_{13}\sigma_\alpha + a_{23}\sigma_\beta + a_{33}\sigma_\gamma + a_{36}\tau_{\alpha\beta} \\
e_{\alpha\beta} = a_{16}\sigma_\alpha + a_{26}\sigma_\beta + a_{36}\sigma_\gamma + a_{66}\tau_{\alpha\beta}
\end{gathered} \tag{1.1}$$

where e_i are the strain components, σ_i and τ_{ik} the stress components, and a_{ik} are elastic constants.

Underlying the theory proposed here is the assumption that the main, from the viewpoint of classical theory, stresses are uniformly distributed over the shell thickness, i.e.

$$\sigma_\alpha = T_1 / h, \quad \sigma_\beta = T_2 / h, \quad \tau_{\alpha\beta} = S / h \tag{1.2}$$

where T_i and S are the internal tangential forces of the shell per unit length of the middle surface.

Taking account of (1.2), we obtain for the moments to $1 \pm k_i\gamma \approx 1$ accuracy

$$M_1 = 0, \quad M_2 = 0, \quad H = 0 \tag{1.3}$$

Henceforth, quantities of the order of $k_i\gamma$ will be neglected as compared with unity to the same accuracy. Certain care is required in discarding terms with factors containing a_{ik}; the fact is that in the general case the elasticity coefficients can form such quantities as cannot, in combination with $k_i\gamma$, be neglected as compared with unity.

2. To the accuracy assumed, the equilibrium equations of a differential element of the body have the form

$$\begin{gathered}
A\,(B\sigma_\alpha)_{,\alpha} - AB_{,\alpha}\sigma_\beta + A\,(A\tau_{\alpha\beta})_{,\beta} + AA_{,\beta}\,\tau_{\beta\alpha} + (H_1^2H_2\tau_{\alpha\gamma})_{,\gamma} = 0 \\
B\,(A\sigma_\beta)_{,\beta} - BA_{,\beta}\sigma_\alpha + B\,(B\tau_{\beta\alpha})_{,\alpha} + BB_{,\alpha}\tau_{\alpha\beta} + (H_2^2H_1\tau_{\beta\gamma})_{,\gamma} = 0 \\
(H_1H_2\sigma_\gamma)_{,\gamma} - \sigma_\alpha ABk_1 - \sigma_\beta ABk_2 + (B\tau_{\alpha\gamma})_{,\alpha} + (A\tau_{\beta\gamma})_{,\beta} = 0 \\
H_1 = A\,(1 + k_1\gamma), \qquad H_2 = B\,(1 + k_2\gamma)
\end{gathered} \tag{2.1}$$

Here H_1, H_2 are Lamé coefficients, A, B the coefficients of the first quadratic form, and k_i the principal curvatures of the middle surface.

Let a shell to be loaded so that on its outer surfaces

$$\tau_{\alpha\gamma} = X^+, \quad \tau_{\beta\gamma} = Y^+, \quad \sigma_\gamma = Z^+, \quad (\gamma = {}^1\!/_2 h) \tag{2.2}$$
$$\tau_{\alpha\gamma} = -X^-, \quad \tau_{\beta\gamma} = -Y^-, \quad \sigma_\gamma = -Z^- \quad (\gamma = -{}^1\!/_2 h)$$

Substituting values of the stresses $\sigma_\alpha, \sigma_\beta, \tau_{\alpha\beta}$ from (1.1) into the two first equilibrium equations (2.1), and integrating with respect to γ, we obtain $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ to the accuracy of the integration functions which should be defined according to (2.2). Substituting the values of $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ obtained here, together with the values of σ_α and σ_β, into the third equilibrium equation and integrating with respect to γ, we obtain σ_γ and still an integration function which should also be defined according to (2.2).

Satisfying the conditions on the surfaces (2.2) by using the values of $\tau_{\alpha\gamma}, \tau_{\beta\gamma}$ and σ_γ obtained above, we have

$$(BT_1)_{,\alpha} - T_2B_{,\alpha} + (AS)_{,\beta} + SA_{,\beta} = -ABX_2$$
$$(AT_2)_{,\beta} - T_1A_{,\beta} + (BS)_{,\alpha} + SB_{,\alpha} = -ABY_2 \tag{2.3}$$
$$T_1k_1 + T_2k_2 = Z_2 + {}^1\!/_2 hZ_1^*$$

$$\tau_{\alpha\gamma} = \frac{X_1}{2} + \frac{\gamma}{h}X_2, \qquad \tau_{\beta\gamma} = \frac{Y_1}{2} + \frac{\gamma}{h}Y_2, \qquad \sigma_\gamma = P_1 + \gamma P_2 + \gamma^2 P_3 \tag{2.4}$$

Here

$$P_1 = \frac{Z_1}{2} + \frac{h}{8}Z_2^*, \quad P_2 = \frac{Z_2}{h}, \quad P_3 = -\frac{1}{2h_i}Z_2^*,$$
$$Z_i^* = \frac{1}{AB}[(BX_i)_{,\alpha} + (AY_i)_{,\beta}]$$
$$X_1 = X^+ - X^-, \quad Y_1 = Y^+ - Y^-, \quad Z_1 = Z^+ - Z^- \tag{2.5}$$
$$X_2 = X^+ + X^-; \quad Y_2 = Y^+ + Y^-, \quad Z_2 = Z^+ + Z^-$$

We therefore obtain the system of equations (2.3), which differs only in the small from the corresponding system of the ordinary theory [1, 4].

Together with (2.3), formulas (2.4] have been found which can be used to determine those stresses which can, in combination with "basic" stresses (1.2), be nominal for shells fabricated from modem reinforced plastics with reduced resistivity to shear and transverse rupture [5].

3. The question of determining the displacements [1, 2] is of special interest in the membrane theory of anisotropic shells.

The relation between the strain and displacement components of any point of the shell $u_\alpha(\alpha, \beta, \gamma)$, $u_\beta(\alpha, \beta, \gamma)$, $u_\gamma(\alpha, \beta, \gamma)$ is written as follows, to the accuracy assumed:

$$e_\alpha = \frac{1}{A}u_{\alpha,\alpha} + \frac{1}{AB}A_{,\beta}u_\beta + k_1u_\gamma \qquad e_\gamma = u_{\gamma,\gamma} \tag{3.1}$$
$$e_\beta = \frac{1}{B}u_{\beta,\beta} + \frac{1}{AB}B_{,\alpha}u_\alpha + k_2u_\gamma$$
$$e_{\beta\gamma} = B\left(\frac{u_\beta}{H_2}\right)_{,\gamma} + \frac{1}{B}u_{\gamma,\beta} \qquad e_{\alpha\beta} = \frac{A}{B}\left(\frac{u_\alpha}{A}\right)_{,\beta} + \frac{B}{A}\left(\frac{u_\beta}{B}\right)_{,\alpha}$$
$$e_{\gamma\alpha} = A\left(\frac{u_\alpha}{H_1}\right)_{,\gamma} + \frac{1}{A}u_{\gamma,\alpha}$$

According to (1.1), (1.2), (2.4) and (3.1) we can write

$$u_{\gamma,\gamma} = a_{13}\frac{T_1}{h} + a_{23}\frac{T_2}{h} + a_{36}\frac{S}{h} + a_{33}(P_1 + \gamma P_2 + \gamma^2 P_3) \qquad (3.2)$$

Integrating (3.2) with respect to γ between the limits 0 and γ, where we assume $u_\gamma = w(\alpha, \beta)$ for $\gamma = 0$, we obtain

$$u_\gamma = w + \gamma\left(a_{13}\frac{T_1}{h} + a_{23}\frac{T_2}{h} + a_{36}\frac{S}{h} + a_{33}P_1\right) + \frac{\gamma^2}{2}a_{33}P_2 + \frac{\gamma^3}{3}a_{33}P_3 \quad (3.3)$$

Here $w(\alpha, \beta)$ is the normal displacement of the shell middle surface.
Furthermore, according to (1.1) and (3.1) we easily obtain, because of (2.4) and (3.3)

$$A\left(\frac{1}{H_1}u_\alpha\right)_{,\gamma} = a_{55}\left(\frac{X_1}{2} + \frac{\gamma}{h}X_2\right) + a_{45}\left(\frac{Y_1}{2} + \frac{\gamma}{h}Y_2\right) - \frac{1}{A}w_{,\alpha} -$$
$$-\frac{\gamma}{A}\left(a_{13}\frac{T_1}{h} + a_{23}\frac{T_2}{h} + a_{36}\frac{S}{h} + a_{33}P_1\right)_{,\alpha} - \frac{\gamma^2}{2A}a_{33}P_{2,\alpha} - \frac{\gamma^3}{3A}a_{33}P_{3,\alpha} \quad (3.4)$$
$$B\left(\frac{1}{H_2}u_\beta\right)_{,\gamma} = a_{44}\left(\frac{Y_1}{2} + \frac{\gamma}{h}Y_2\right) + a_{45}\left(\frac{X_1}{2} + \frac{\gamma}{h}X_2\right) - \frac{1}{B}w_{,\beta} -$$
$$-\frac{\gamma}{B}\left(a_{13}\frac{T_1}{h} + a_{23}\frac{T_2}{h} + a_{36}\frac{S}{h} + a_{33}P_1\right)_{,\beta} - \frac{\gamma^2}{2B}a_{33}P_{2,\beta} - \frac{\gamma^3}{3B}a_{33}P_{3,\beta}$$

Integrating (3.4) with respect to γ within the limits 0 and γ, and assuming here that $u_\alpha = u(\alpha, \beta)$, $u_\beta = v(\alpha, \beta)$ for $\gamma = 0$, we obtain for the tangential displacements of any point of the shell

$$u_\alpha = u - \frac{\gamma}{A}w_{,\alpha} + a_{55}\left(\frac{\gamma}{2}X_1 + \frac{\gamma^2}{2h}X_2\right) + a_{45}\left(\frac{\gamma}{2}Y_1 + \frac{\gamma^2}{2h}Y_2\right) -$$
$$-\frac{\gamma^2}{2A}P_{1,\alpha}{}^* - \frac{\gamma^3}{6A}a_{33}P_{2,\alpha} - \frac{\gamma^4}{12A}a_{33}P_{3,\alpha} \qquad (3.5)$$
$$u_\beta = v - \frac{\gamma}{B}w_{,\beta} + a_{44}\left(\frac{\gamma}{2}Y_1 + \frac{\gamma^2}{2h}Y_2\right) + a_{45}\left(\frac{\gamma}{2}X_1 + \frac{\gamma^2}{2h}X_2\right) -$$
$$-\frac{\gamma^2}{2B}P^*_{1,\beta} - \frac{\gamma^3}{6B}a_{33}P_{2,\beta} - \frac{\gamma^4}{12B}a_{33}P_{3,\beta}$$

Substituting the values of u_α, u_β, u_γ from (3.5) and (3.3), respectively, into the geometric relationships (3.1) which have still not been utilized fully, we obtain

$$e_\alpha = \varepsilon_1 + \gamma\left[L_1(w) + \frac{1}{2A}F_{1,\alpha} + \frac{1}{2AB}A_{,\beta}Q_1 + k_1P_1{}^*\right] +$$
$$+\gamma^2\left[\frac{1}{2}L_1(P_1{}^*) - \frac{1}{2Ah}F_{2,\alpha} + \frac{1}{2ABh}A_{,\beta}Q_2 + k_1\frac{a_{33}}{2}P_2\right] +$$
$$+\gamma^3\left[\frac{a_{33}}{6}L_1(P_2) + k_1\frac{a_{33}}{3}P_3\right] + \gamma^4\frac{a_{33}}{12}L_1(P_3)$$
$$e_\beta = \varepsilon_2 + \gamma\left[L_2(w) + \frac{1}{2B}Q_{1,\beta} + \frac{1}{2AB}B_{,\alpha}F_1 + k_2P_1{}^*\right] + \qquad (3.6)$$
$$+\gamma^2\left[\frac{1}{2}L_2(P_1{}^*) + \frac{1}{2Bh}Q_{2,\beta} + \frac{1}{2ABh}B_{,\alpha}F_2 + k_2\frac{a_{33}}{2}P_2\right] +$$
$$+\gamma^3\left[\frac{a_{33}}{6}L_2(P_2) + k_2\frac{a_{33}}{3}P_3\right] + \gamma^4\frac{a_{33}}{12}L(P_3)$$
$$e_{\alpha\beta} = \omega + \gamma\left[2L_3(w) + \frac{A}{2B}\left(\frac{F_1}{A}\right)_{,\beta} + \frac{B}{2A}\left(\frac{Q_1}{B}\right)_{,\alpha}\right] +$$
$$+\gamma^2\left[L_3(P_1{}^*) + \frac{A}{2Bh}\left(\frac{F_2}{A}\right)_{,\beta} + \frac{B}{2Ah}\left(\frac{Q_2}{B}\right)_{,\alpha}\right] +$$
$$+\gamma^3\frac{a_{33}}{3}L_3(P_2) + \gamma^4\frac{a_{33}}{6}L_3(P_3)$$

where the following notation has been used

$$F_i = a_{55}X_i + a_{45}Y_i, \qquad Q_i = a_{44}Y_i + a_{45}X_i \tag{3.7}$$

$$P_1{}^* = a_{13}\frac{T_1}{h} + a_{23}\frac{T_2}{h} + a_{36}\frac{S}{h} + a_{33}P_1 = T^* + a_{33}P_1$$

$$T^* = a_{13}\frac{T_1}{h} + a_{23}\frac{T_2}{h} + a_{36}\frac{S}{h}$$

as well as the known representations

$$\varepsilon_1 = \frac{1}{A}u_{,\alpha} + \frac{1}{AB}A_{,\beta}v + k_1 w, \qquad \varepsilon_2 = \frac{1}{B}v_{,\beta} + \frac{1}{AB}B_{,\alpha}u + k_2 w \tag{3.8}$$

$$\omega = \frac{A}{B}\left(\frac{u}{A}\right)_{,\beta} + \frac{B}{A}\left(\frac{v}{B}\right)_{,\alpha}$$

For the linear operators L_i we have

$$L_1(q) = -\frac{1}{A}\left(\frac{1}{A}q_{,\alpha}\right)_{,\alpha} - \frac{1}{AB^2}A_{,\beta}q_{,\beta}, \tag{3.9}$$

$$L_2(q) = -\frac{1}{B}\left(\frac{1}{B}q_{,\beta}\right)_{,\beta} - \frac{1}{A^2B}B_{,\alpha}q_{,\alpha}$$

$$L_3(q) = -\frac{1}{AB}\left(q_{,\alpha\beta} - \frac{1}{A}A_{,\beta}q_{,\alpha} - \frac{1}{B}B_{,\alpha}q_{,\beta}\right)$$

On the other hand, according to (1.2), (2.4), (2.5), (3.7), for the generalized Hooke's law equations needed here, we have

$$e_\alpha = a_{11}\frac{T_1}{h} + a_{12}\frac{T_2}{h} + a_{16}\frac{S}{h} + a_{13}\left(\frac{Z_1}{2} + \frac{h}{8}Z_2{}^*\right) + \gamma a_{13}\frac{Z_2}{h} - \gamma^2\frac{a_{13}}{2h}Z_2{}^*$$

$$e_\beta = a_{22}\frac{T_2}{h} + a_{12}\frac{T_1}{h} + a_{26}\frac{S}{h} + a_{23}\left(\frac{Z_1}{2} + \frac{h}{8}Z_2{}^*\right) + \gamma a_{23}\frac{Z_2}{h} - \gamma^2\frac{a_{23}}{2h}Z_2{}^*$$

$$e_{\alpha\beta} = a_{66}\frac{S}{h} + a_{16}\frac{T_1}{h} + a_{26}\frac{T_2}{h} + a_{36}\left(\frac{Z_1}{2} + \frac{h}{8}Z_2{}^*\right) + \gamma a_{36}\frac{Z_2}{h} - \gamma^2\frac{a_{36}}{2h}Z_2{}^*$$

In the expressions presented for the strain components (see (3.6) also), the factors for the γ of all powers can be so small in some particular cases that to the accuracy accepted they should be neglected as compared with ε_i, ω or as compared with terms containing γ to lower degree. However, it is impossible to say or do this in the general case. Hence, in the subsequent discussions all the terms in the formulas for the components of the strain tensor must be left without any modification; however, terms having the order $k_i\gamma$ in obvious cases can hence be neglected as compared with unity.

Comparing the values of the strain components obtained with the corresponding representations (3.6), and equating coefficients of γ to the zero power, we obtain

$$\frac{1}{A}u_{,\alpha} + \frac{1}{AB}A_{,\beta}v + k_1 w = a_{11}\frac{T_1}{h} + a_{12}\frac{T_2}{h} + a_{16}\frac{S}{h} + \frac{a_{13}}{2}Z_1 + a_{13}\frac{h}{8}Z_2{}^* \tag{3.10}$$

$$\frac{1}{B}v_{,\beta} + \frac{1}{AB}B_{,\alpha}u + k_2 w = a_{22}\frac{T_2}{h} + a_{12}\frac{T_1}{h} + a_{26}\frac{S}{h} + \frac{a_{23}}{2}Z_1 + a_{23}\frac{h}{8}Z_2{}^*$$

$$\frac{A}{B}\left(\frac{u}{A}\right)_{,\beta} + \frac{B}{A}\left(\frac{v}{B}\right)_{,\alpha} = a_{66}\frac{S}{h} + a_{16}\frac{T_1}{h} + a_{26}\frac{T_2}{h} + \frac{a_{36}}{2}Z_1 + a_{36}\frac{h}{8}Z_2{}^*$$

The equations presented are a complete system of differential equations which can be utilized to determine the desired displacements of the problem. These equations contain terms which appeared as a result of taking account of phenomena associated with the transverse deformations of the shell. The effect of these terms on the displacements is probably not always perceptible; however, these terms must be left for analysis in the general case because cases are not excluded in which taking account of the phenomena related to the transverse deformations of the shell can become necessary.

Furthermore, equating coefficients of the remaining γ, we obtain four groups of relationships.

First group

$$L_1(w) + k_1 T^* = \frac{a_{13}}{h} Z_2 - a_{33}\frac{h}{8} L_1(Z_2) - k_1\frac{a_{33}}{2} Z_1 - \frac{1}{2A} F_{1,\alpha} - \frac{1}{2AB} A_{,\beta} Q_1$$

$$L_2(w) + k_2 T^* = \frac{a_{23}}{h} Z_2 - a_{33}\frac{h}{8} L_2(Z_2) - k_2\frac{a_{33}}{2} Z_1 - \frac{1}{2B} Q_{1,\beta} - \frac{1}{2AB} B_{,\alpha} F_1$$

$$2L_3(w) = \frac{a_{36}}{h} Z_2 - \frac{1}{2}\frac{B}{A}\left(\frac{1}{B} Q_1\right)_{,\alpha} - \frac{1}{2}\frac{A}{B}\left(\frac{1}{A} F_1\right)_{,\beta} \qquad (3.11)$$

Second group

$$L_1(T^*) = -\frac{a_{33}}{2} L_1(Z_1) - k_1\frac{a_{33}}{h} Z_2 - \frac{a_{13}}{h} Z_2^* - \frac{1}{Ah} F_{2,\alpha} - \frac{1}{ABh} A_{,\beta} Q_2$$

$$L_2(T^*) = -\frac{a_{33}}{2} L_2(Z_1) - k_2\frac{a_{33}}{h} Z_2 - \frac{a_{23}}{h} Z_2^* - \frac{1}{Bh} Q_{2,\beta} - \frac{1}{ABh} B_{,\alpha} F_2$$

$$L_3(T^*) = -\frac{a_{33}}{2} L_3(Z_1) - \frac{a_{33}}{2h} Z_2^* - \frac{1}{2h}\frac{A}{B}\left(\frac{1}{A} F_2\right)_{,\beta} - \frac{1}{2h}\frac{B}{A}\left(\frac{1}{B} Q_2\right)_{,\alpha} \qquad (3.12)$$

Third group

$$L_1(Z_2) - k_1 Z_2^* = 0, \quad L_2(Z_2) - k_2 Z_2^* = 0, \quad L_3(Z_2) = 0 \qquad (3.13)$$

Fourth group

$$L_1(Z_2^*) = 0, \qquad L_2(Z_2^*) = 0, \qquad L_3(Z_2^*) = 0 \qquad (3.14)$$

Satisfying conditions (3.11)–(3.14), we completely assure a membrane shell state. The degree of purity of the membrane state of the shell depends on the level of accuracy of compliance with conditions (3.11)–(3.14).

Examining (3.11)–(3.14), we note also that constraints should be imposed not only on the shell geometry and the external loading to assure a membrane state of the shell, but also on the mechanical characteristics of the material of the shell. More accurately, consistent geometric, static and physical constraints should be imposed on the shell.

4. The stresses σ_α, σ_β and $\tau_{\alpha\beta}$, on which the hypothetical constraints were imposed, can be represented by using strains.

According to (2.4) and (3.6) we obtain from (1.1)

$$a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{16}\tau_{\alpha\beta} = \varepsilon_1 - a_{13}\left(\frac{Z_1}{2} + \frac{h}{8} Z_2^*\right) +$$

$$+ \gamma\left[L_1(w) + \frac{1}{2A} F_{1,\alpha} + \frac{1}{2AB} A_{,\beta} Q_1 + k_1 P_1^* - a_{13}\frac{Z_2}{h}\right] +$$

$$+ \gamma^2\left[\frac{1}{2} L_1(P_1^*) + \frac{1}{2Ah} F_{2,\alpha} + \frac{1}{2ABh} A_{,\beta} Q_2 + k_1\frac{a_{33}}{2} P_2 + \right. \qquad (4.1)$$

$$\left. + \frac{a_{13}}{2h} Z_2^*\right] + \gamma^3\left[\frac{a_{33}}{6h} L_1(Z_2) - k_1\frac{a_{33}}{6h} Z_2^*\right] - \gamma^4\frac{a_{33}}{24h} L_1(Z_2^*)$$

$$a_{22}\sigma_\beta + a_{12}\sigma_\alpha + a_{26}\tau_{\alpha\beta} = \varepsilon_2 - a_{23}\left(\frac{Z_1}{2} + \frac{h}{8}Z_2^*\right) +$$
$$+ \gamma\left[L_2(w) + \frac{1}{2B}Q_{1,\beta} + \frac{1}{2AB}B_{,\alpha}F_1 + k_2P_1^* - a_{23}\frac{Z_2}{h}\right] + \qquad (4.2)$$
$$+ \gamma^2\left[\frac{1}{2}L_2(P_1^*) + \frac{1}{2Bh}Q_{2,\beta} + \frac{1}{2ABh}B_{,\alpha}F_2 + k_2\frac{a_{33}}{2}P_2 + \frac{a_{23}}{2h}Z_2^*\right] +$$
$$+ \gamma^3\left[\frac{a_{33}}{6h}L_2(Z_2) - k_2\frac{a_{33}}{6h}Z_2^*\right] - \gamma^4\frac{a_{33}}{24h}L_2(Z_2^*)$$

$$a_{66}\tau_{\alpha\beta} + a_{16}\sigma_\alpha + a_{26}\sigma_\beta = \omega - a_{36}\left(\frac{Z_1}{2} + \frac{h}{8}Z_2^*\right) +$$
$$+ \gamma\left[2L_3(w) + \frac{A}{2B}\left(\frac{F_1}{A}\right)_{,\beta} + \frac{B}{2A}\left(\frac{Q_1}{B}\right)_{,\alpha} - a_{36}\frac{Z_2}{h}\right] + \qquad (4.3)$$
$$+ \gamma^2\left[L_3(P_1^*) + \frac{A}{2Bh}\left(\frac{F_2}{A}\right)_{,\beta} + \frac{B}{2Ah}\left(\frac{Q_2}{B}\right)_{,\alpha} + a_{36}\frac{Z_2^*}{2h}\right] +$$
$$+ \gamma^3\frac{a_{33}}{3h}L_3(Z_2) - \gamma^4\frac{a_{33}}{12h}L_3(Z_2^*)$$

According to (3.11)–(3.14), and taking account of (2.4), (2.6) and (3.7), we obtain from (4.1)–(4.3) to determine the stresses

$$a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{16}\tau_{\alpha\beta} = \varepsilon_1 - a_{13}P_1$$
$$a_{12}\sigma_\alpha + a_{22}\sigma_\beta + a_{26}\tau_{\alpha\beta} = \varepsilon_2 - a_{23}P_1$$
$$a_{16}\sigma_\alpha + a_{26}\sigma_\beta + a_{66}\tau_{\alpha\beta} = \omega - a_{36}P_1 \qquad (4.4)$$

Solving the system (4.4) for the stresses, we finally find (see also (3.8) for representations of ε_i, ω)

$$\sigma_\alpha = B_{11}\varepsilon_1 + B_{12}\varepsilon_2 + B_{16}\omega - K_1\left\{\frac{Z_1}{2} + \frac{h}{8}\frac{1}{AB}[(BX_2)_{,\alpha} + (AY_2)_{,\beta}]\right\}$$
$$\sigma_\beta = B_{22}\varepsilon_2 + B_{12}\varepsilon_1 + B_{26}\omega - K_2\left\{\frac{Z_1}{2} + \frac{h}{8}\frac{1}{AB}[(BX_2)_{,\alpha} + (AY_2)_{,\beta}]\right\} \qquad (4.5)$$
$$\tau_{\alpha\beta} = B_{66}\omega + B_{16}\varepsilon_1 + B_{26}\varepsilon_2 - K_6\left\{\frac{Z_1}{2} + \frac{h}{8}\frac{1}{AB}[(BX_2)_{,\alpha} + (AY_2)_{,\beta}]\right\}$$

$$K_i = B_{i1}a_{13} + B_{i2}a_{23} + B_{i6}a_{36} \qquad (4.6)$$

$$B_{11} = (a_{22}a_{66} - a_{26}^2)\,\Omega^{-1}, \qquad B_{16} = (a_{12}a_{26} - a_{22}a_{16})\,\Omega^{-1}$$
$$B_{22} = (a_{11}a_{66} - a_{16}^2)\,\Omega^{-1}, \qquad B_{26} = (a_{12}a_{16} - a_{11}a_{26})\,\Omega^{-1}$$
$$B_{66} = (a_{11}a_{22} - a_{12}^2)\,\Omega^{-1}, \quad B_{12} = (a_{16}a_{26} - a_{12}a_{66})\,\Omega^{-1}$$
$$\Omega = (a_{11}a_{22} - a_{12}^2)\,a_{66} + 2a_{12}a_{16}a_{26} - a_{11}a_{26}^2 - a_{22}a_{16}^2 \qquad (4.7)$$

Examining (4.5), we remark that the stresses σ_α, σ_β, $\tau_{\alpha\beta}$ obtained in this manner do not vary over the shell thickness, i.e. the initial condition that the shell be in the membrane state is assured.

It is seen from the means of obtaining the formulas for the stresses elucidated above, that the conditions (3.11)–(3.14) are sufficient to assure a membrane state of stress of the shell.

However, conditions (3.11)–(3.14), which assure the membrane state of the shell, can be replaced by weakened conditions assuring the membrane state of the shell to some approximation.

The weakened conditions can be obtained by assuming that those parts of the stresses

σ_α, σ_β, $\tau_{\alpha\beta}$, which vary over the shell thickness are negligibly small as compared with those parts of the stresses which do not vary with thickness.

The weakened conditions for the membrane state are easily obtained from (4.1)-(4.3)

$$\Big|\Big\{\gamma\left[L_1(w)+\frac{1}{2A}F_{1,\alpha}+\frac{1}{2AB}A_{,\beta}Q_1+k_1P_1^*-a_{13}\frac{Z_2}{h}\right]+$$
$$+\gamma^2\left[\frac{1}{2}L_1(P_1^*)+\frac{1}{2Ah}F_{2,\alpha}+\frac{1}{2ABh}A_{,\beta}Q_2+k_1\frac{a_{33}}{2}P_2+\right.$$
$$\left.+\frac{a_{13}}{2h}Z_2^*\right]+\gamma^3\left[\frac{a_{33}}{6h}L_1(Z_2)-k_1\frac{a_{33}}{6h}Z_2^*\right]-$$
$$-\gamma^4\frac{a_{33}}{24h}L_1(Z_2^*)\Big\}[\varepsilon_1-a_{13}P_1]^{-1}\Big|\leqslant|(k_ih)^m|$$

$$\Big|\Big\{\gamma\left[L_2(w)+\frac{1}{2B}Q_{1,\beta}+\frac{1}{2AB}B_{,\alpha}F_1+k_2P_1^*-a_{23}\frac{Z_2}{h}\right]+$$
$$+\gamma^2\left[\frac{1}{2}L_2(P_1^*)+\frac{1}{2Bh}Q_{2,\beta}+\frac{1}{2ABh}B_{,\alpha}F_2+k_2\frac{a_{33}}{2}P_2+\right.$$
$$\left.+\frac{a_{23}}{2h}Z_2^*\right]+\gamma^3\left[\frac{a_{33}}{6h}L_2(Z_2)-k_2\frac{a_{33}}{6h}Z_2^*\right]-$$
$$-\gamma^4\frac{a_{33}}{24h}L_2(Z_2^*)\Big\}[\varepsilon_2-a_{23}P_1]^{-1}\Big|\leqslant|(k_ih)^m|$$

$$\Big|\Big\{\gamma\left[2L_3(w)+\frac{A}{2B}\left(\frac{F_1}{A}\right)_{,\beta}+\frac{B}{2A}\left(\frac{Q_1}{B}\right)_{,\alpha}-a_{36}\frac{Z_2}{h}\right]+$$
$$+\gamma^2\left[L_3(P_1^*)+\frac{A}{2Bh}\left(\frac{F_2}{A}\right)_{,\beta}+\frac{B}{2Ah}\left(\frac{Q_2}{B}\right)_{,\alpha}+a_{36}\frac{Z_2^*}{2h}\right]+$$
$$+\gamma^3\frac{a_{33}}{3h}L_3(Z_2)-\gamma^4\frac{a_{33}}{12h}L_3(Z^*)\Big\}[\omega-a_{36}P_1]^{-1}\Big|\leqslant|(k_ih)^m| \tag{4.8}$$

where m is a number characterizing the degree of the accuracy required.

5. The membrane theory of anisotropic shells under the assumption that the normal displacement u_γ is independent of the coordinate γ merits attention.

To the fundamental assumption (1.2) let us append a new assumption which is represented analytically as follows:

$$e_\gamma=u_{\gamma,\gamma}=0, \qquad u_\gamma=w(\alpha,\ \beta) \tag{5.1}$$

In the new formulation the static part of the problem does not undergo any changes, i.e. the internal forces will be defined in conformity with (2.3), and the stresses by means of (1.2), (2.4), (4.5). The formulas and equations for the displacements are simplified considerably.

Proceeding as usual, we obtain for the tangential displacements

$$u_\alpha=u-\frac{\gamma}{A}w_{,\alpha}+a_{55}\left(\frac{\gamma}{2}X_1+\frac{\gamma^2}{2h}X_2\right)+a_{45}\left(\frac{\gamma}{2}Y_1+\frac{\gamma^2}{2h}Y_2\right) \tag{5.2}$$
$$u_\beta=v-\frac{\gamma}{A}w_{,\beta}+a_{44}\left(\frac{\gamma}{2}Y_1+\frac{\gamma^2}{2h}Y_2\right)+a_{45}\left(\frac{\gamma}{2}X_1+\frac{\gamma^2}{2h}X_2\right)$$

Furthermore, for the strain components still not utilized we obtain the following more simplified representations

$$e_\alpha=\varepsilon_1+\gamma\left[L_1(w)+\frac{1}{2A}F_{1,\alpha}+\frac{1}{2AB}A_{,\beta}Q_1\right]+\gamma^2\left[\frac{1}{2Ah}F_{2,\alpha}+\frac{1}{2ABh}A_{,\beta}Q_2\right]$$
$$e_\beta=\varepsilon_2+\gamma\left[L_2(w)+\frac{1}{2B}Q_{1,\beta}+\frac{1}{2AB}B_{,\alpha}F\right]+ \tag{5.3}$$

$$+\gamma^2\left[\frac{1}{2Bh}Q_{2,\beta}+\frac{1}{2ABh}B_{,\alpha}F_2\right]$$

$$e_{\alpha\beta}=\omega+\gamma\left[L_3(w)+\frac{A}{2B}\left(\frac{F_1}{A}\right)_{,\beta}+\frac{B}{2A}\left(\frac{Q_1}{B}\right)_{,\alpha}\right]+$$
$$+\gamma^2\left[\frac{A}{2Bh}\left(\frac{F_2}{A}\right)_{,\beta}+\frac{B}{2Ah}\left(\frac{Q_2}{B}\right)_{,\alpha}\right]$$

Comparing (5.2) and (5.3) with (3.5) and (3.6), respectively, we note the essential simplification in the new representations, which is that there are no longer any terms with factors for γ greater than the second degree here. The coefficients of lower powers of γ are also simplified somewhat.

Taking account of (1.2), (2.4)–(2.6) and (3.7), in conformity with (1.1) and (5.3) we obtain three equations to determine the desired displacements and two groups of relations, compliance with which will assure the membrane state of the shell.

The equations to determine the displacements are no different than the equations of the general case obtained earlier, i.e. than (3.10).

However, by taking (5.1) we thereby weaken the conditions (3.11)–(3.14) somewhat, since in place of four groups of relationships for the membrane state we here obtain two simpler groups of relationships.

First group

$$\begin{aligned}
L_1(w)&=a_{13}\frac{Z_2}{h}-\frac{1}{2A}F_{1,\alpha}-\frac{1}{2AB}A_{,\beta}Q_1\\
L_2(w)&=a_{23}\frac{Z_2}{h}-\frac{1}{2B}Q_{1,\beta}-\frac{1}{2AB}B_{,\alpha}F_1\\
2L_3(w)&=a_{36}\frac{Z_2}{h}-\frac{1}{2}\frac{B}{A}\left(\frac{Q_1}{B}\right)_{,\alpha}-\frac{1}{2}\frac{A}{B}\left(\frac{F_1}{A}\right)_{,\beta}
\end{aligned}\qquad(5.4)$$

Second group

$$\begin{aligned}
&\frac{1}{A}F_{2,\alpha}+\frac{1}{AB}A_{,\beta}Q_2=-a_{13}Z_2^*\\
&\frac{1}{B}Q_{2,\beta}+\frac{1}{AB}B_{,\alpha}F_2=-a_{23}Z_2^*\\
&\frac{A}{B}\left(\frac{F_2}{A}\right)_{,\beta}+\frac{B}{A}\left(\frac{Q_2}{B}\right)_{,\alpha}=-a_{36}Z_2^*
\end{aligned}\qquad(5.5)$$

The weakened membrane state conditions are also simplified, where we have in place of (4.8)

$$\left|\left\{\gamma\left[L_1(w)+\frac{1}{2A}F_{1,\alpha}+\frac{1}{2AB}A_{,\beta}Q_1-a_{13}\frac{Z_2}{h}\right]+\gamma^2\left[\frac{a_{13}}{2h}Z_2^*+\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{2Ah}F_{2,\alpha}+\frac{1}{2ABh}A_{,\beta}Q_2\right]\right\}[\varepsilon_1-a_{13}P]^{-1}\right|\leqslant|(k_ih)^m|$$

$$\left|\left\{\gamma\left[L_2(w)+\frac{1}{2B}Q_{1,\beta}+\frac{1}{2AB}B_{,\alpha}F_1-a_{23}\frac{Z_2}{h}\right]+\gamma^2\left[\frac{a_{23}}{2h}Z_2^*+\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{2Bh}Q_{2,\beta}+\frac{1}{2ABh}B_{,\alpha}F_2\right]\right\}[\varepsilon_2-a_{23}P_1]^{-1}\right|\leqslant|(k_ih)^m|\qquad(5.6)$$

$$\left|\left\{\gamma\left[2L_3(w)+\frac{A}{2B}\left(\frac{F_1}{A}\right)_{,\beta}+\frac{B}{2A}\left(\frac{Q_1}{B}\right)_{,\alpha}-a_{36}\frac{Z_2}{h}\right]+\right.\right.$$
$$\left.\left.+\gamma^2\left[\frac{A}{2Bh}\left(\frac{F_2}{A}\right)_{,\beta}+\frac{B}{2Ah}\left(\frac{Q_2}{B}\right)_{,\alpha}+a_{36}\frac{Z_2^*}{2h}\right]\right\}[\omega-a_{36}P_1]^{-1}\right|\leqslant|(k_ih)^m|$$

6. As an illustration, let us consider a primitive example.

A closed circular cylindrical shell ($k_1 = 0$, $k_2 = 1/R_2 = 1/R$, $A = 1$, $B = 1$, length l) is clamped at the endfaces perpendicular to the axis of rotation so that

$$u = 0, \quad v = 0 \quad (\alpha = 0)$$
$$T_1 = 0, \quad S = 0 \quad (\alpha = l) \tag{6.1}$$

The shell is loaded so that

$$X^+ = p\alpha, \quad X^- = 0, \quad Z^+ = -q, \quad Z^- = 0, \quad Y^+ = Y^- = 0 \tag{6.2}$$

then we obtain from (2.4)–(2.6) for the load terms

$$X_1 = X_2 = p\alpha, \quad Z_1 = Z_2 = -q, \quad Y_1 = Y_2 = 0 \tag{6.3}$$
$$Z_1{}^* = Z_2{}^* = p, \quad P_1 = {}^1\!/_8 hp - {}^1\!/_2 q, \quad P_2 = -{}^1\!/_2 q, \quad P_3 = -{}^1\!/_2 p / h$$

Solving the system (2.3) taking account of (6.1)–(6.3), we obtain for the tangential forces

$$T_1 = {}^1\!/_2 p\,(l^2 - \alpha^2), \quad S = 0, \quad T_2 = R\,({}^1\!/_2 hp - q) \tag{6.4}$$

By virtue of (6.2)–(6.4) we have from (1.2), (2.4) for the stresses

$$\sigma\alpha = \frac{p}{2h}(l^2 - \alpha^2), \quad \sigma_\beta = \frac{R}{h}\left(\frac{h}{2}p - q\right), \quad \tau_{\alpha\beta} = 0 \tag{6.5}$$
$$\tau_{\alpha\gamma} = p\alpha\left(\frac{1}{2} + \frac{\gamma}{h}\right), \quad \tau_{\beta\gamma} = 0, \quad \sigma_\gamma = \left(\frac{h}{8}p - \frac{q}{2}\right) - \gamma\frac{q}{2} - \gamma^2\frac{p}{2h}$$

All these stresses are evidently of nominal interest for an anisotropic shell.

Solving the system (3.10) taking account of (6.1), (6.3), (6.4), we obtain for the displacements $u(\alpha)$ $v(\alpha)$, $w(\alpha)$

$$u = a_{11}\frac{P}{2h}\left(l\alpha^2 - \frac{\alpha^3}{3}\right) + a_{12}\frac{R}{h}\left(\frac{h}{2}p - q\right)\alpha - a_{13}\left(\frac{q}{2} - \frac{h}{8}p\right)\alpha \tag{6.6}$$
$$v = a_{16}\frac{P}{2h}\left(l\alpha^2 - \frac{\alpha^3}{3}\right) + a_{26}\frac{R}{h}\left(\frac{h}{2}p - q\right)\alpha - a_{36}\left(\frac{q}{2} - \frac{h}{8}p\right)\alpha$$
$$w = a_{12}\frac{PR}{2h}(l^2 - \alpha^2) + a_{22}\frac{R^2}{h}\left(\frac{h}{2}p - q\right) - a_{23}R\left(\frac{q}{2} - \frac{h}{8}p\right)$$

Examining (6.5) and (6.6), we note that nominal stresses and displacements can appear in a shell in the new membrane theory formulation, which occur due to taking account of the phenomena associated with the transverse mechanical characteristics of the shell.

In the case under consideration, the membrane state conditions (6.5) and (5.5) are written as follows in conformity with (3.7), (3.9)(6.2), (6.3):

$$-a_{12}\frac{pR}{h} + a_{55}\frac{p}{2} + a_{13}\frac{q}{h} = 0, \quad a_{23}\frac{q}{h} = 0, \quad a_{33}\frac{q}{h} + a_{45}\frac{p}{2} = 0 \tag{6.7}$$
$$a_{55}p + a_{13}p = 0, \quad a_{23}p = 0, \quad a_{45}p + a_{36}p = 0 \tag{6.8}$$

From (6.7) and (6.8) those values of the geometric, force and physical characteristics of the shell for which the membrane state of the shell will be assured are easily established.

For example, let $p = 0$, $q \neq 0$; conditions (6.7) and (6.8) become

$$a_{13}\frac{q}{h} = 0, \quad a_{23}\frac{q}{h} = 0, \quad a_{36}\frac{q}{h} = 0 \tag{6.9}$$

Then in order to assure a membrane state it is sufficient that $a_{13} = a_{23} = a_{36} = 0$ (the condition $h \to \infty$ is not discussed for obvious reasons). In this case, from (6.6) we obtain for the displacements

$$u = -a_{12}\frac{qR}{h}\alpha, \qquad v = -a_{26}\frac{qR}{h}\alpha, \qquad w = -a_{22}\frac{qR^2}{h} \tag{6.10}$$

The values of the displacements (6.10) presented here agree with the appropriate displacements obtained by classical theory [1], just as it should. Assuming $a_{13} = a_{23} = a_{36} = 0$, we pointedly neglect the normal stresses σ_γ while determining the displacements, and in combination with the initial assumptions we accepted above; this forms the complex of initial hypotheses of the classical theory.

Finally, let us mention that the weakened membrane state conditions (5.6) are satisfied in the example discussed $(p = 0,\ q \neq 0)$ if, for example $a_{13} / a_{12} < 1$, $a_{23} / a_{22} < 1$, $a_{36} / a_{26} < 1$ (for $m = 1$).

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ON ELASTICITY RELATIONSHIPS IN THE LINEAR THEORY OF THIN ELASTIC SHELLS

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Questions of constructing tensor elsticity relationships within the accuracy of the Kirchhoff-Love hypotheses are discussed herein. It is clarified that it is impossible to conserve simultaneously the static-geometric analogy and to assure application of theorems of the theory of elasticity by using not too complex elasticity relationships. In this connection, two modifications of the elasticity relationships in the linear theory of thin elastic shells are proposed. The first modification retains these theorems in the linear theory of thin elastic shells. The second modification satisfies the requirements of the static-geometric analogy, but violates the reciprocity theorem (in the small).

Among the possible modifications of the elasticity relationships used in the linear theory of thin elastic shells, one of the most simple ones is the modification presented by the authors of [1] and [2]. Nevertheless, these relationships answer a number of requirements to be discussed below. At the same time the authors of [3] indicated that these relationships are not of tensor character.

In this paper we consider the construction of tensor elasticity relationships differing slightly in the lines of curvature from relationships presented in [1] and [2].